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Exact propagators in harmonic superspace

Sergei M. Kuzenko

School of Physics, The University of Western Australia
Crawley, W.A. 6009, Australia
kuzenko@physics.uwa.edu.au

Abstract

Within the background field formulation in harmonic superspace for quantum $\mathcal{N}=2$ super Yang-Mills theories, the propagators of the matter, gauge and ghost superfields possess a complicated dependence on the SU(2) harmonic variables via the background vector multiplet. This dependence is shown to simplify drastically in the case of an on-shell vector multiplet. For a covariantly constant background vector multiplet, we exactly compute all the propagators. In conjunction with the covariant multi-loop scheme developed in hep-th/0302205, these results provide an efficient (manifestly $\mathcal{N}=2$ supersymmetric) technical setup for computing multi-loop quantum corrections to effective actions in $\mathcal{N}=2$ supersymmetric gauge theories, including the $\mathcal{N}=4$ super Yang-Mills theory.

Within the background field formulation for quantum $\mathcal{N}=2$ super Yang-Mills theories [1, 2, 3] in harmonic superspace[1] [4, 5, 6], there occur three types of background-dependent Green's functions: (i) the q-hypermultiplet propagator; (ii) the ω-hypermultiplet propagator (corresponding, in particular, to the Faddeev-Popov ghosts); (iii) the vector multiplet propagator. Compared with the background-field propagators in ordinary gauge theories or in $\mathcal{N}=1$ super Yang-Mills theories, the $\mathcal{N}=2$ superpropagators have a much more complicated structure, due to a nontrivial dependence on SU(2) harmonics, the internal space variables, which enter even the corresponding covariant d'Alembertian [1]. As is shown below, the harmonic dependence of the $\mathcal{N}=2$ superpropagators simplifies drastically if the background vector multiplet satisfies classical equations of motion. Using these observations makes it possible to compute the exact propagators, in a straightforward way, in the presence of a covariantly constant background vector multiplet.

We start by assembling the necessary information about the $\mathcal{N}=2$ Yang-Mills supermultiplet which is known to possess an off-shell formulation [7] in conventional $\mathcal{N}=2$ superspace $\mathbb{R}^{4|8}$ parametrized by coordinates $z^A=(x^a,\theta^\alpha_i,\bar\theta^i_{\dot\alpha})$, where $i=\underline{1},\underline{2}$. The gauge covariant derivatives are defined by

$$\mathcal{D}_A=(\mathcal{D}_a,\mathcal{D}^i_\alpha,\bar{\mathcal{D}}^{\dot\alpha}_i)=D_A+\mathrm{i}\,\Gamma_A(z)\ , \tag{1}$$

with $D_A=(D_a,D^i_\alpha,\bar D^{\dot\alpha}_i)$ the flat covariant derivatives, and Γ_A the gauge connection. Their gauge transformation law is

$$\mathcal{D}_A\ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\,\mathcal{D}_A\,\mathrm{e}^{-\mathrm{i}\tau(z)}\ ,\qquad \tau^\dagger=\tau\ , \tag{2}$$

with the gauge parameter $\tau(z)$ being arbitrary modulo the reality condition imposed. The gauge covariant derivatives obey the algebra [7]:

$$\begin{aligned}
&\{\mathcal{D}^i_\alpha,\bar{\mathcal{D}}_{\dot\alpha j}\}=-2\mathrm{i}\,\delta^i_j\mathcal{D}_{\alpha\dot\alpha}\\
&\{\mathcal{D}^i_\alpha,\mathcal{D}^j_\beta\}=2\mathrm{i}\,\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{\mathcal{W}}\ ,\qquad \{\bar{\mathcal{D}}_{\dot\alpha i},\bar{\mathcal{D}}_{\dot\beta j}\}=2\mathrm{i}\,\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{ij}\mathcal{W}\ ,\\
&\big[\mathcal{D}^i_\alpha,\mathcal{D}_{\beta\dot\beta}\big]=\varepsilon_{\alpha\beta}\bar{\mathcal{D}}^i_{\dot\beta}\bar{\mathcal{W}}\ ,\qquad \big[\bar{\mathcal{D}}_{\dot\alpha i},\mathcal{D}_{\beta\dot\beta}\big]=\varepsilon_{\dot\alpha\dot\beta}\mathcal{D}_{\beta i}\mathcal{W}\ ,\\
&\big[\mathcal{D}_{\alpha\dot\alpha},\mathcal{D}_{\beta\dot\beta}\big]=\mathrm{i}\,\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}=\frac{\mathrm{i}}{4}\,\varepsilon_{\dot\alpha\dot\beta}\,\mathcal{D}^i{}_{(\alpha}\mathcal{D}_{\beta)i}\mathcal{W}-\frac{\mathrm{i}}{4}\,\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}^i{}_{(\dot\alpha}\bar{\mathcal{D}}_{\dot\beta)i}\bar{\mathcal{W}}\ .
\end{aligned} \tag{3}$$

The superfield strengths $\mathcal{W}$ and $\bar{\mathcal{W}}$ satisfy the Bianchi identities

$$\bar{\mathcal{D}}^i_{\dot\alpha}\mathcal{W}=\mathcal{D}^i_\alpha\bar{\mathcal{W}}=0\ ,\qquad \mathcal{D}^{ij}\mathcal{W}=\bar{\mathcal{D}}^{ij}\bar{\mathcal{W}}\ , \tag{4}$$

[1] The Feynman rules in $\mathcal{N}=2$ harmonic superspace were developed in [5, 21].

where

$$\mathcal{D}^{ij} = \mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_{\alpha} \ , \qquad \bar{\mathcal{D}}^{ij} = \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{j)\dot\alpha} \ . \tag{5}$$

The $\mathcal{N}=2$ harmonic superspace $\mathbb{R}^{4|8}\times S^2$ [4, 5, 6] extends conventional superspace by the two-sphere $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ parametrized by harmonics, i.e., group elements

$$(u_i{}^-\,,\,u_i{}^+)\in\mathrm{SU}(2)\ , \quad u^+_i = \varepsilon_{ij}u^{+j}\ , \quad \overline{u^{+i}} = u^-_i\ , \quad u^{+i}u^-_i = 1\ . \tag{6}$$

In harmonic superspace, both the $\mathcal{N}=2$ Yang-Mills supermultiplet and hypermultiplets can be described by *unconstrained* superfields over the analytic subspace of $\mathbb{R}^{4|8}\times S^2$ parametrized by the variables $\xi \equiv (y^a, \theta^{+\alpha}, \bar\theta^+_{\dot\alpha}, u^+_i, u^-_j)$, where the so-called analytic basis is defined by

$$y^a = x^a - 2\mathrm{i}\,\theta^{(i}\sigma^a\bar\theta^{j)}u^+_i u^-_j\ , \qquad \theta^\pm_\alpha = u^\pm_i\theta^i_\alpha\ , \qquad \bar\theta^\pm_{\dot\alpha} = u^\pm_i\bar\theta^i_{\dot\alpha}\ . \tag{7}$$

With the notation

$$\mathcal{D}^\pm_\alpha = u^\pm_i\mathcal{D}^i_\alpha\ , \qquad \bar{\mathcal{D}}^\pm_{\dot\alpha} = u^\pm_i\bar{\mathcal{D}}^i_{\dot\alpha}\ , \tag{8}$$

it follows from (3) that the operators $\mathcal{D}^+_\alpha$ and $\bar{\mathcal{D}}^+_{\dot\alpha}$ strictly anticommute, $\{\mathcal{D}^+_\alpha,\mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot\alpha},\bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha,\bar{\mathcal{D}}^+_{\dot\alpha}\} = 0$. A covariantly analytic superfield $\Phi^{(p)}(z,u)$ is defined to obey the constraints

$$\mathcal{D}^+_\alpha\Phi^{(p)} = \bar{\mathcal{D}}^+_{\dot\alpha}\Phi^{(p)} = 0\ . \tag{9}$$

Here the superscript p refers to the harmonic U(1) charge, $\mathcal{D}^0\,\Phi^{(p)} = p\,\Phi^{(p)}$, where $\mathcal{D}^0$ is one of the harmonic gauge covariant derivatives[2]

$$\mathcal{D}^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\ , \qquad \mathcal{D}^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\ . \tag{10}$$

The operator $\mathcal{D}^{++}$ acts on the space of covariantly analytic superfields.

In the framework of the background field formalism in $\mathcal{N}=2$ harmonic superspace [1, 2, 3], there appear three types of covariant (matter and gauge field) propagators:

$$\begin{aligned}
G^{(1,1)}(z,u\,,z',u') &= \frac{1}{\widehat{\Box}}\,(\mathcal{D}^+)^4\,(\mathcal{D}'^+)^4\left\{\mathbf{1}\,\delta^{12}(z-z')\,\frac{1}{(u^+u'^+)^3}\right\}\ ,\\
G^{(0,0)}(z,u\,,z',u') &= -\frac{1}{\widehat{\Box}}\,(\mathcal{D}^+)^4\,(\mathcal{D}'^+)^4\left\{\mathbf{1}\,\delta^{12}(z-z')\,\frac{(u^-u'^-)}{(u^+u'^+)^3}\right\}\ ,\\
G^{(2,2)}(z,u\,,z',u') &= -\frac{1}{\widehat{\Box}}\,(\mathcal{D}^+)^4\Big\{\mathbf{1}\,\delta^{12}(z-z')\,\delta^{(-2,2)}(u,u')\Big\}\ .
\end{aligned} \tag{11}$$

[2]Throughout this paper, we use the so-called τ-frame [4, 6] defined in the appendix.

Here $\widehat{\Box}$ is the analytic d'Alembertian [1],

$$\widehat{\Box} = \mathcal{D}^a\mathcal{D}_a - \frac{\mathrm{i}}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha - \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} + \frac{\mathrm{i}}{4}(\mathcal{D}^{+\alpha}\mathcal{D}^+_\alpha\mathcal{W})\mathcal{D}^{--}$$
$$-\frac{\mathrm{i}}{8}[\mathcal{D}^{+\alpha},\mathcal{D}^-_\alpha]\mathcal{W} - \frac{1}{2}\{\bar{\mathcal{W}},\mathcal{W}\}\ , \tag{12}$$

$\delta^{12}(z-z')$ denotes the $\mathcal{N}=2$ superspace delta-function,

$$\delta^{12}(z-z') = \delta^4(x-x')\,(\theta-\theta')^4(\bar\theta-\bar\theta')^4\ , \tag{13}$$

$(u^+u'^+)^{-3}$ and $(u^-u'^-)(u^+u'^+)^{-3}$ denote special harmonic distributions [5, 6], and finally the two-point function $\delta^{(p,-p)}(u,u')$, with $p\in\mathbb{Z}$, stands for a harmonic delta-function [5, 6]. The Green's function $G^{(1,1)}$ determines the Feynman propagator of a covariantly analytic matter superfield q^+ (the so-called q-hypermultiplet [4, 6]) transforming in some representation R of the gauge group,

$$\mathrm{i}\,\langle q^+(z,u)\,\breve{q}^+(z',u')\rangle = G^{(1,1)}(z,u\,\boldsymbol{,}\,z',u')\ , \tag{14}$$

with $\breve{q}^+$ the analyticity-preserving conjugate of q^+. The Green's function $G^{(0,0)}$ determines the Feynman propagator of a covariantly analytic matter (or, in the case of the adjoint representation, ghost) superfield ω (the so-called ω-hypermultiplet [4, 6]) in the representations R of the gauge group,

$$\mathrm{i}\,\langle \omega(z,u)\,\breve{\omega}(z',u')\rangle = G^{(0,0)}(z,u\,\boldsymbol{,}\,z',u')\ , \tag{15}$$

When R coincides with the adjoint representation, the Green's function $G^{(2,2)}$ determines the Feynman propagator of the quantum gauge superfield,

$$\mathrm{i}\,\langle v^{++}(z,u)\,v^{++\mathrm{T}}(z',u')\rangle = G^{(2,2)}(z,u\,\boldsymbol{,}\,z',u')\ , \tag{16}$$

see [1, 2, 3] for more details. In what follows, we will study in detail only the Green's functions $G^{(1,1)}$ and $G^{(2,2)}$, since the structure of $G^{(0,0)}$ is very similar to that of $G^{(1,1)}$.

The operator $\widehat{\Box}$ acts on the space of covariantly analytic superfields. Given such a superfield $\Phi^{(p)}$ obeying the constraints (9), one can check

$$\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2\,\Phi^{(p)} = \widehat{\Box}\,\Phi^{(p)}\ . \tag{17}$$

Among the important properties of $\widehat{\Box}$ is the following [2]:

$$(\mathcal{D}^+)^4\,\widehat{\Box} = \widehat{\Box}\,(\mathcal{D}^+)^4\ . \tag{18}$$

In addition, for an arbitrary superfield $V^{(p)}(z,u)$ of U(1) charge p, we have

$$[\mathcal{D}^{++},\widehat{\Box}]\,V^{(p)} = \frac{\mathrm{i}}{2}\Big\{\frac{1}{2}(p-1)\,(\mathcal{D}^+\mathcal{D}^+\mathcal{W}) - (\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^+_\alpha - (\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{+\dot\alpha}\Big\}V^{(p)}~. \qquad (19)$$

Eq. (18) implies, in particular, that $G^{(1,1)}$ and $G^{(2,2)}$ can be rewritten as follows:

$$\begin{aligned} G^{(1,1)}(z,u\,,z',u') &= \quad(\mathcal{D}^+)^4\,(\mathcal{D}'^+)^4\,\frac{1}{\widehat{\Box}}\,\Big\{\mathbf{1}\,\delta^{12}(z-z')\,\frac{1}{(u^+u'^+)^3}\Big\}~, \\ G^{(2,2)}(z,u\,,z',u') &= -(\mathcal{D}^+)^4\,\frac{1}{\widehat{\Box}}\,\Big\{\mathbf{1}\,\delta^{12}(z-z')\,\delta^{(-2,2)}(u,u')\Big\}~. \end{aligned} \qquad (20)$$

Sometimes, it is useful to rewrite $G^{(2,2)}$ in a manifestly analytic form, following [6],

$$G^{(2,2)}(z,u\,,z',u') = -\frac{1}{2\,\widehat{\Box}^2}\,(\mathcal{D}^+)^4(\mathcal{D}'^+)^4\Big\{\mathbf{1}\,\delta^{12}(z-z')\,(\mathcal{D}^{--})^2\delta^{(2,-2)}(u,u')\Big\}~. \qquad (21)$$

This representation may be, in principle, advantageous when handling those supergraphs which contain a product of harmonic distributions and require the introduction of a harmonic regularization at intermediate stages of the calculation.

Let us introduce a new second-order operator Δ,

$$\Delta = \widehat{\Box} + \frac{\mathrm{i}}{2}(\mathcal{D}^{-\alpha}\mathcal{W})\mathcal{D}^+_\alpha + \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^-_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{+\dot\alpha}~, \qquad (22)$$

which coincides with $\widehat{\Box}$ on the space of covariantly analytic superfields,

$$\mathcal{D}^+_\alpha\Phi^{(p)} = \bar{\mathcal{D}}^+_{\dot\alpha}\Phi^{(p)} = 0 \quad\Longrightarrow\quad \Delta\,\Phi^{(p)} = \widehat{\Box}\,\Phi^{(p)}~. \qquad (23)$$

In terms of the operator introduced, the property (18) turns into

$$(\mathcal{D}^+)^4\Big\{\Delta + \frac{\mathrm{i}}{2}\,[\mathcal{D}^{+\alpha},\mathcal{D}^-_\alpha]\mathcal{W}\Big\} = \Delta\,(\mathcal{D}^+)^4~, \qquad (24)$$

while eq. (19) turns into

$$[\mathcal{D}^{++},\Delta]\,V^{(p)} = \frac{\mathrm{i}}{4}(p-1)\,(\mathcal{D}^+\mathcal{D}^+\mathcal{W})V^{(p)}~. \qquad (25)$$

If the background vector multiplet satisfies the classical equations of motion,

$$\mathcal{D}^{ij}\mathcal{W} = \bar{\mathcal{D}}^{ij}\bar{\mathcal{W}} = 0~, \qquad (26)$$

then the operator Δ is characterized by the following two properties:

$$(\mathcal{D}^+)^4\,\Delta = \Delta\,(\mathcal{D}^+)^4~, \qquad [\mathcal{D}^{++},\Delta] = 0~. \qquad (27)$$

The latter is equivalent to the fact that Δ is harmonic-independent,[3]

$$\Delta = \mathcal{D}^a \mathcal{D}_a + \frac{\mathrm{i}}{2}(\mathcal{D}^{\alpha}_i \mathcal{W})\mathcal{D}^i_{\alpha} - \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^i_{\dot\alpha} \bar{\mathcal{W}})\bar{\mathcal{D}}^{\dot\alpha}_i - \frac{1}{2}\{\bar{\mathcal{W}}, \mathcal{W}\} \ , \tag{28}$$

while it maps the space of covariantly anaytic superfields into itself. Taking into account the properties of Δ described, the Green's functions $G^{(1,1)}$ and $G^{(2,2)}$ can be rewritten in the form:

$$\begin{aligned} G^{(1,1)}(z,u\,\mathbf{,}\,z',u') &= \quad (\mathcal{D}^+)^4\,(\mathcal{D}'^+)^4\,\frac{1}{\Delta}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\}\,\frac{1}{(u^+u'^+)^3}\ , \\ G^{(2,2)}(z,u\,\mathbf{,}\,z',u') &= -(\mathcal{D}^+)^4\,\frac{1}{\Delta}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\}\,\delta^{(-2,2)}(u,u') \\ &= -(\mathcal{D}^+)^4(\mathcal{D}'^+)^4\,\frac{1}{2\Delta^2}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\}\,(\mathcal{D}^{--})^2\,\delta^{(2,-2)}(u,u')\ . \end{aligned} \tag{29}$$

The dependence of $G^{(1,1)}$ and $G^{(2,2)}$ on the harmonic is completely factorized.[4] At this stage, it is advantageous to introduce, following Fock and Schwinger, the proper-time representation:

$$\begin{aligned} \frac{1}{\Delta^n}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\} &= \frac{(-\mathrm{i})^n}{(n-1)!}\int_0^\infty \mathrm{d}s\,s^{n-1}\,K(z,z'|s)\,\mathrm{e}^{-\varepsilon\,s}\ , \qquad \varepsilon\to+0 \\ K(z,z'|s) &= \mathrm{e}^{\mathrm{i}s\Delta}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\}\ , \end{aligned} \tag{30}$$

with $n=1$ or 2.

A further simplification occurs in the case of a covariantly constant vector multiplet,

$$\mathcal{D}_a\mathcal{W} = \mathcal{D}_a\bar{\mathcal{W}} = 0 \quad \Longrightarrow \quad [\mathcal{W},\bar{\mathcal{W}}] = 0\ . \tag{31}$$

Then, the first-order operator appearing in (28),

$$\Upsilon = \frac{\mathrm{i}}{2}(\mathcal{D}^{\alpha}_i \mathcal{W})\mathcal{D}^i_{\alpha} - \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^i_{\dot\alpha} \bar{\mathcal{W}})\bar{\mathcal{D}}^{\dot\alpha}_i - \bar{\mathcal{W}}\mathcal{W}\ , \tag{32}$$

turns out to commute with the vector covariant derivative,

$$[\Upsilon, \mathcal{D}_a] = 0\ , \tag{33}$$

and this is similar to the $\mathcal{N}=1$ case [9]. As in the $\mathcal{N}=1$ case, we can now associate with the heat kernel $K(z,z'|s)$, eq. (30), a reduced kernel $\tilde{K}(z,z'|s)$. The latter is defined as follows:

$$K(z,z'|s) = = \mathrm{e}^{\mathrm{i}s\Upsilon}\,\mathrm{e}^{\mathrm{i}s\mathcal{D}^a\mathcal{D}_a}\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\} = \mathrm{e}^{\mathrm{i}s\Upsilon}\,\tilde{K}(z,z'|s)\ . \tag{34}$$

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[3] The operator Δ could have been introduced several years ago in [3].

[4] This technical result is actually of utmost importance. The point is that it allows us to keep the harmonic dependence of $\mathcal{N}=2$ supergraphs under control. Some supergraphs may involve potentially dangerous coinciding harmonic singularities [6, 3, 8] which have to be treated extremely carefully.

The reduced heat kernel $\tilde{K}(z,z'|s)$ can now be evaluated in the same way as it was done in the $\mathcal{N}=1$ superspace case [10, 9] by generalizing the Schwinger construction [11]. The resullt is

$$\tilde{K}(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\det{}^{1/2}\left(\frac{s\mathcal{F}}{\sinh(s\mathcal{F})}\right)\,\mathrm{e}^{\frac{\mathrm{i}}{4}\rho^a(\mathcal{F}\coth(s\mathcal{F}))_{ab}\rho^b}\,\zeta^4\,\bar{\zeta}^4\,I(z,z')\ , \tag{35}$$

where the determinant is computed with respect to the Lorentz indices, and

$$\zeta^4 = (\theta-\theta')^4\ , \qquad \bar{\zeta}^4 = (\bar{\theta}-\bar{\theta}')^4\ . \tag{36}$$

Here we have introduced the $\mathcal{N}=2$ supersymmetric interval $\zeta^A \equiv \zeta^A(z,z') = -\zeta^A(z',z)$ defined by

$$\zeta^A = \begin{cases} \rho^a = (x-x')^a - \mathrm{i}(\theta-\theta')_i\sigma^a\bar{\theta}'^i + \mathrm{i}\theta'_i\sigma^a(\bar{\theta}-\bar{\theta}')^i\ , \\ \zeta_i^\alpha = (\theta-\theta')_i^\alpha\ , \\ \bar{\zeta}^i_{\dot\alpha} = (\bar{\theta}-\bar{\theta}')^i_{\dot\alpha}\ . \end{cases} \tag{37}$$

The parallel displacement propagator, $I(z,z')$, and its properties [9] are collected in the appendix. The reduced kernel turns out to solve the evolution problem

$$\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s} + \mathcal{D}^a\mathcal{D}_a\right)\tilde{K}(z,z'|s) = 0\ , \qquad \tilde{K}(z,z'|0) = \delta^{12}(z-z') = \delta^4(\rho)\,\zeta^4\,\bar{\zeta}^4 \tag{38}$$

as a consequence, in particular, of eq. (A.3) and of the identity

$$\zeta^4\,\bar{\zeta}^4\,\mathcal{D}_a I(z,z') = -\,\frac{\mathrm{i}}{2}\,\zeta^4\,\bar{\zeta}^4\,\mathcal{F}_{ab}\,\rho^b I(z,z')\ , \tag{39}$$

which the parallel displacement propagator obeys, see the appendix.

In order to obtain the complete kernel $K(z,z'|s)$, one still has to evaluate the action of the operator $\exp(\mathrm{i}s\Upsilon)$ on the reduced kernel $\tilde{K}(z,z'|s)$. This is accomplished in complete analogy with the $\mathcal{N}=1$ case [9]

$$\begin{aligned} K(z,z'|s) &= -\frac{\mathrm{i}}{(4\pi s)^2}\,\det{}^{1/2}\left(\frac{s\mathcal{F}}{\sinh(s\mathcal{F})}\right)\,\mathrm{e}^{\frac{\mathrm{i}}{4}\rho(s)\mathcal{F}\coth(s\mathcal{F})\rho(s)}\,\zeta^4(s)\,\bar{\zeta}^4(s)\,\mathrm{e}^{\mathrm{i}s\Upsilon}I(z,z')\ , \\ \zeta^A(s) &= \mathrm{e}^{\mathrm{i}s\Upsilon}\,\zeta^A\,\mathrm{e}^{-\mathrm{i}s\Upsilon}\ . \end{aligned} \tag{40}$$

The components of $\zeta^A(s) = (\rho^a(s), \zeta_i^\alpha(s), \bar{\zeta}^i_{\dot\alpha}(s))$ can be easily evaluated using the (anti) commutation relations (3) and the obvious identity

$$\mathcal{D}_B\,\zeta^A = \delta_B{}^A + \frac{1}{2}\,\zeta^C\,T_{CB}{}^A\ , \tag{41}$$

with $T_{CB}{}^{A}$ the flat superspace torsion. The action of $\exp(\mathrm{i}s\Upsilon)$ on the parallel displacement propagator $I(z,z')$ is evaluated using the relation (A.8) or (A.9); the result is quite lengthy and is not reproduced here.

Making use of the (anti)commutation relations for the covariant derivatives, eq. (3), one can check that

$$\big[\Delta, \mathcal{D}^i_\alpha\big] = -\frac{\mathrm{i}}{2}\,\mathcal{D}^j{}_{(\alpha}\mathcal{D}_{\beta)j}\mathcal{W}\,\mathcal{D}^{\beta i}~, \qquad \big[\Delta, \bar{\mathcal{D}}^i_{\dot\alpha}\big] = \frac{\mathrm{i}}{2}\,\bar{\mathcal{D}}^j{}_{(\dot\alpha}\bar{\mathcal{D}}_{\dot\beta)j}\bar{\mathcal{W}}\,\bar{\mathcal{D}}^{\dot\beta i}~, \tag{42}$$

and therefore

$$\big[\Delta, \mathcal{D}^{ij}\big] = 0~, \qquad \big[\Delta, \bar{\mathcal{D}}^{ij}\big] = 0~. \tag{43}$$

The latter identities imply the following important properties:

$$\mathcal{D}_{ij}\,K(z,z'|s) = \mathcal{D}'_{ij}\,K(z,z'|s)~, \qquad \bar{\mathcal{D}}_{ij}\,K(z,z'|s) = \bar{\mathcal{D}}'_{ij}\,K(z,z'|s)~. \tag{44}$$

Since

$$(\mathcal{D}^+)^4 = \frac{1}{16}\,\mathcal{D}^{ij}\,\bar{\mathcal{D}}^{kl}\,u^+_i u^+_j u^+_k u^+_l = \frac{1}{16}\,\bar{\mathcal{D}}^{ij}\,\mathcal{D}^{kl}\,u^+_i u^+_j u^+_k u^+_l~, \tag{45}$$

the relations (44) allow one to accumulate the overall D-factors, which occur in

$$(\mathcal{D}^+)^4(\mathcal{D}'^+)^4\,K(z,z'|s) \tag{46}$$

at different superspace points, z and z', to a single point, say, z. One then obtains

$$\begin{aligned}(\mathcal{D}^+)^4(\mathcal{D}'^+)^4\,K(z,z'|s) = (\mathcal{D}^+)^4\Big\{&(u^+u'^+)^4\,(\mathcal{D}^-)^4 - \frac{\mathrm{i}}{2}(u^+u'^+)^3(u^-u'^+)\,\Omega^{--} \\ &+(u^+u'^+)^2(u^-u'^+)^2\,\Delta\Big\}K(z,z'|s)~,\end{aligned} \tag{47}$$

where

$$\Omega^{--} = \mathcal{D}^{\alpha\dot\alpha}\mathcal{D}^-_\alpha\bar{\mathcal{D}}^-_{\dot\alpha} + \frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2 + \frac{1}{2}\bar{\mathcal{W}}(\bar{\mathcal{D}}^-)^2 + (\mathcal{D}^-\mathcal{W})\mathcal{D}^- + (\bar{\mathcal{D}}^-\bar{\mathcal{W}})\bar{\mathcal{D}}^-~. \tag{48}$$

The relation (47) is analogous to the representation for the two-point function

$$(\mathcal{D}^+)^4(\mathcal{D}'^+)^4\Big\{\mathbf{1}\,\delta^{12}(z-z')\Big\} \tag{49}$$

derived in [12]. The last term in eq. (47) can be rewritten in an equivalent form by taking into account the fact that $K(z,z'|s)$ satisfies the evolution equation

$$\Big(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s}\;+\;\Delta\Big)\,K(z,z'|s)\;=\;0~. \tag{50}$$

It also follows from eq. (44) and the obvious property of harmonic delta functions

$$u_i^+ u_j^+ u_k^+ u_l^+ \, \delta^{(-2,2)}(u,u') = u_i'^+ u_j'^+ u_k'^+ u_l'^+ \, \delta^{(2,-2)}(u,u') \tag{51}$$

that

$$(\mathcal{D}^+)^4 \Big\{ K(z,z'|s) \, \delta^{(-2,2)}(u,u') \Big\} = (\mathcal{D}'^+)^4 \Big\{ K(z,z'|s) \, \delta^{(2,-2)}(u,u') \Big\} \; . \tag{52}$$

The latter guarantees that the gauge field Green's function $G^{(2,2)}(z,u;z',u')$ is covariantly analytic in both arguments.

The action of a product of covariant derivatives on the kernel $K(z,z'|s)$ can be readily evaluated using the algebra of covariant derivatives (3) and the fundamental relations (A.8) and (A.9).

In conclusion, we would like to list several interesting open problems that can be addressed using the approach advocated in this note. First of all, it becomes now feasible to compute, in a manifestly $\mathcal{N}=2$ supersymmetric way, multi-loop quantum corrections to the effective action for a non-Abelian vector multiplet which is only constrained to be on-shell but otherwise arbitrary. Indeed, the representation (29) implies that the harmonic dependence of the propagators is completely factorized (this dependence is actually of the same form possessed by the free propagators), and therefore under control. As regards the superfield heat kernel $K(z,z'|s)$, eq. (30), its manifestly $\mathcal{N}=2$ supersymmetric and gauge covariant derivative expansion can be developed in complete analogy with the $\mathcal{N}=1$ case worked out in [9]. Another application is the calculation of low-energy effective actions for $\mathcal{N}=2,4$ super Yang-Mills theories on the Coulomb branch, including $\mathcal{N}=2$ supersymmetric Heisenberg-Euler[5] type actions [14]. Calculation of multi-loop quantum corrections to such actions can be carried out with the use of the exact propagators, in the presence of a covariantly constant vector multiplet, which have been found in this note.

It has recently been demonstrated [15] (see also [13] and references therein) that for a self-dual background the two-loop QED effective action takes a remarkably simple form that is very similar to the one-loop action in the same background. There are expectations that this similarity persists at higher loops, and therefore there should be some remarkable structure encoded in the all-loop effective action for gauge theories. In the supersymmetric case, one has to replace the requirement of self-duality by that of relaxed super self-duality [16] in order to arrive at conclusions similar to those given in [15]. Further progress in this direction may be achieved through the analysis of $\mathcal{N}=2$ covariant supergraphs.

[5]See [13] for a recent review of Heisenberg-Euler effective Lagrangians.

Finally, we believe that the results of this note may be helpful in the context of the conjectured correspondence [17, 18, 19] between the D3-brane action in $AdS_5 \times S^5$ and the low-energy action for $\mathcal{N} = 4$ SU(N) SYM on its Coulomb branch, with the gauge group SU(N) spontaneously broken to SU($N-1$)$\times$U(1). There have appeared two independent F^6 tests of this conjecture [19, 20], with conflicting conclusions. The approach advocated here provides the opportunity for a further test.

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Appendix A Parallel displacement propagator

In this appendix we describe, basically following [9], the main properties of the parallel displacement propagator $I(z,z')$ in $\mathcal{N} = 2$ superspace. This object is uniquely specified by the following requirements:
(i) the gauge transformation law

$$I(z,z') \;\to\; \mathrm{e}^{\mathrm{i}\tau(z)}\, I(z,z')\, \mathrm{e}^{-\mathrm{i}\tau(z')} \tag{A.1}$$

with respect to the gauge (τ-frame) transformation of the covariant derivatives (2);
(ii) the equation

$$\zeta^A \mathcal{D}_A\, I(z,z') = \zeta^A \Big(D_A + \mathrm{i}\,\Gamma_A(z)\Big) I(z,z') = 0 \;; \tag{A.2}$$

(iii) the boundary condition

$$I(z,z) = \mathbf{1} \;. \tag{A.3}$$

These imply the important relation

$$I(z,z')\, I(z',z) = \mathbf{1} \;, \tag{A.4}$$

as well as

$$\zeta^A \mathcal{D}'_A\, I(z,z') = \zeta^A \Big(D'_A\, I(z,z') - \mathrm{i}\, I(z,z')\, \Gamma_A(z')\Big) = 0 \;. \tag{A.5}$$

Let $\Psi(z)$ be a *harmonic-independent* superfield transforming in some representation of the gauge group,

$$\Psi(z) \;\to\; \mathrm{e}^{\mathrm{i}\tau(z)}\,\Psi(z)\ . \tag{A.6}$$

Then it can be represented by the covariant Taylor series [9]

$$\Psi(z) = I(z,z')\,\sum_{n=0}^{\infty}\frac{1}{n!}\,\zeta^{A_n}\ldots\zeta^{A_1}\,\mathcal{D}'_{A_1}\ldots\mathcal{D}'_{A_n}\,\Psi(z')\ . \tag{A.7}$$

The fundametal properties of the parallel displacement propagator are [9]

$$\begin{aligned}\mathcal{D}_B I(z,z') = \mathrm{i}\,I(z,z')\,\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\Big\{n\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}'_{A_1}\ldots\mathcal{D}'_{A_{n-1}}\mathcal{F}_{A_n\,B}(z')\\ +\frac{1}{2}(n-1)\,\zeta^{A_n}T_{A_n\,B}{}^{C}\,\zeta^{A_{n-1}}\ldots\zeta^{A_1}\mathcal{D}'_{A_1}\ldots\mathcal{D}'_{A_{n-2}}\mathcal{F}_{A_{n-1}\,C}(z')\Big\}\ ,\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}\mathcal{D}_B I(z,z') = \mathrm{i}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\Big\{-\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-1}}\mathcal{F}_{A_n\,B}(z)\\ +\frac{1}{2}(n-1)\,\zeta^{A_n}T_{A_n\,B}{}^{C}\,\zeta^{A_{n-1}}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-2}}\mathcal{F}_{A_{n-1}\,C}(z)\Big\}\,I(z,z')\ .\end{aligned} \tag{A.9}$$

Here $\mathcal{F}_{AB}$ denotes the superfield strength defined as follows

$$[\mathcal{D}_A,\mathcal{D}_B\} = T_{AB}{}^{C}\mathcal{D}_C + \mathrm{i}\,\mathcal{F}_{AB}\ . \tag{A.10}$$

In the case of a covariantly constant vector multiplet, the series in (A.8) and (A.9) terminate as only the tensors $\mathcal{F}_{CD}$, $\mathcal{D}_A\mathcal{F}_{CD}$ and $\mathcal{D}_A\mathcal{D}_B\mathcal{F}_{CD}$ have non-vanishing components.

Throughout this paper, we work in the τ-frame, in which the gauge covariant derivatives $\mathcal{D}_A$ are harmonic-independent, although the harmonic superspace practitioners often use the so-called λ-frame [4, 6]. To go over to the λ-frame, one has to transform

$$\mathcal{D}_A \quad\longrightarrow\quad \mathrm{e}^{\mathrm{i}\Omega}\,\mathcal{D}_A\,\mathrm{e}^{-\mathrm{i}\Omega}\ , \tag{A.11}$$

and similarly for matter superfields, where $\Omega(z,u)$ is the bridge superfield [4, 6] which occurs as follows

$$\mathcal{D}^+_\alpha = \mathrm{e}^{-\mathrm{i}\Omega}\,D^+_\alpha\,\mathrm{e}^{\mathrm{i}\Omega}\ ,\qquad \bar{\mathcal{D}}^+_\alpha = \mathrm{e}^{-\mathrm{i}\Omega}\,\bar{D}^+_\alpha\,\mathrm{e}^{\mathrm{i}\Omega}\ . \tag{A.12}$$

In the λ-frame, the gauge covariant derivative $\mathcal{D}^+_a$ and $\bar{\mathcal{D}}^+_{\dot\alpha}$ coincide with the flat derivatives D^+_α and $\bar{D}^+_{\dot\alpha}$, respectively. The parallel displacement propagator in the λ-frame is obtained from that in the τ-frame by the transformation

$$I(z,z') \equiv I_\tau(z,z') \quad\longrightarrow\quad \mathrm{e}^{\mathrm{i}\Omega(z,u)}\,I(z,z')\,\mathrm{e}^{-\mathrm{i}\Omega(z',u')} \equiv I_\lambda(z,u\,,z',u')\ . \tag{A.13}$$

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